Chapter 25W

Meshes

25.1W The cotangent Laplacian

We mentioned the cotangent Laplacian in passing in the main book. Here we give just a little more detail.

The ***cotangent Laplacian*** is defined at a vertex $\mathbf{v}$ in terms of its neighbors, which we'll denote $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, For a typical vertex $\mathbf{v}_j$, we denote by α_j and β_j the angles $\mathbf{v}\mathbf{v}_{j-1}\mathbf{v}_j$ and $\mathbf{v}\mathbf{v}_{j+1}\mathbf{v}_j$, as in Figure 25W.1. (As usual, we treat $\mathbf{v}_{n+1}$ as meaning $\mathbf{v}_1$.) The cotangent Laplacian at $\mathbf{v}$ is defined by

$$\sum_{i=1}^{n} (\cot \alpha_j + \cot \beta_j)(\mathbf{v}_j - \mathbf{v}).$$

It's derived by taking a weighted sum of the vectors from $\mathbf{v}$ to each $\mathbf{v}_j$, where the weights are proportional to the areas of wedges of the Voronoi diagram, which is where the cotangents arise.

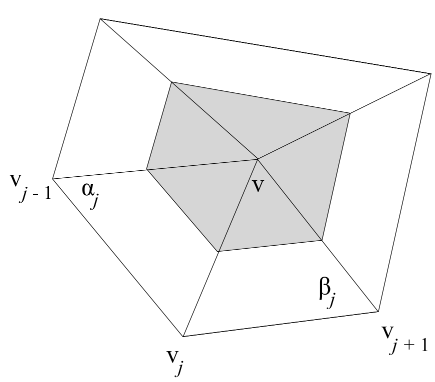

Figure 25W.1: A vertex $\mathbf{v}$*, its Voronoi neighborhood (shaded), a vertex* $\mathbf{v}_j$ *in its 1-ring, and the angles* α_j *and* β_j*.*

The critical feature of this formulation is that if the star of $\mathbf{v}$ is *planar*, then $L(\mathbf{v}) = \mathbf{0}$ for the cotangent Laplacian (in contrast to the uniform Laplacian). More important, however, is that when it is scaled down by the area $A(\mathbf{v})$ of the Voronoi region around $\mathbf{v}$ (i.e., the set of points of the mesh whose on-the-mesh distance to $\mathbf{v}$ is smaller than to any other vertex), the result, $\frac{1}{4A(\mathbf{v})}L(\mathbf{v})$ is the product of the discrete mean curvature $\bar{\kappa}$ at $\mathbf{v}$ and the normal $\mathbf{n}$ at $\mathbf{v}$, where the discrete mean curvature is the mesh analog of the mean curvature for smooth surfaces, and the direction of the cotangent Laplacian can be taken as *defining* a normal vector at the vertex, although we must contend with the possibility that the cotangent Laplacian will be zero, and with the problem that the direction vector may flip from outward-pointing to inward pointing as we move from regions of positive curvature to regions of negative curvature.

Many more geometric computations can be performed in the discrete setting; Meyer et al.[1] provide a fine introduction.

1. Mark Meyer, Mathieu Desbrun, Peter Schröder, and Alan H. Barr. *Discrete Differential-Geometry Operators for Triangulated 2-Manifolds*, in Hans-Christian Hege and Konrad Polthier (Eds.), *Visualization and Mathematics III*, pages 35–57. Springer-Verlag, Heidelberg, 2003.